

# Two distance-regular graphs

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2011/06/11

## Abstract

We construct two families of distance-regular graphs, namely the subgraph of the dual polar graph of type  $B_3(q)$  induced on the vertices far from a fixed point, and the subgraph of the dual polar graph of type  $D_4(q)$  induced on the vertices far from a fixed edge. The latter is the extended bipartite double of the former.

## 1 The extended bipartite double

We shall use  $\sim$  to indicate adjacency in a graph. For notation and definitions of concepts related to distance-regular graphs, see [BCN]. We repeat the definition of extended bipartite double.

The *bipartite double* of a graph  $\Gamma$  with vertex set  $X$  is the graph with vertex set  $\{x^+, x^- \mid x \in X\}$  and adjacencies  $x^\delta \sim y^\epsilon$  iff  $\delta\epsilon = -1$  and  $x \sim y$ . The bipartite double of a graph  $\Gamma$  is bipartite, and it is connected iff  $\Gamma$  is connected and not bipartite. If  $\Gamma$  has spectrum  $\Phi$ , then its bipartite double has spectrum  $(-\Phi) \cup \Phi$ . See also [BCN], Theorem 1.11.1.

The *extended bipartite double* of a graph  $\Gamma$  with vertex set  $X$  is the graph with vertex set  $\{x^+, x^- \mid x \in X\}$ , and the same adjacencies as the bipartite double, except that also  $x^- \sim x^+$  for all  $x \in X$ . The extended bipartite double of a graph  $\Gamma$  is bipartite, and it is connected iff  $\Gamma$  is connected. If  $\Gamma$  has spectrum  $\Phi$ , then its extended bipartite double has spectrum  $(-\Phi - 1) \cup (\Phi + 1)$ . See also [BCN], Theorem 1.11.2.

## 2 Far from an edge in the dual polar graph of type $D_4(q)$

Let  $V$  be a vector space of dimension 8 over a field  $F$ , provided with a nondegenerate quadratic form of maximal Witt index. The maximal totally isotropic subspaces of  $V$  (of dimension 4) fall into two families  $\mathcal{F}_1$  and  $\mathcal{F}_2$ , where the dimension of the intersection of two elements of the same family is even (4 or 2 or 0) and the dimension of the intersection of two elements of different families is odd (3 or 1).

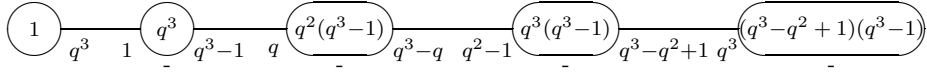
The geometry of the totally isotropic subspaces of  $V$ , where  $A \in \mathcal{F}_1$  and  $B \in \mathcal{F}_2$  are incident when  $\dim A \cap B = 3$  and otherwise incidence is symmetrized inclusion, is known as the geometry  $D_4(F)$ . The bipartite incidence graph on

the maximal totally isotropic subspaces is known as the dual polar graph of type  $D_4(F)$ .

Below we take  $F = \mathbf{F}_q$ , the finite field with  $q$  elements, so that graph and geometry are finite. We shall use projective terminology, so that 1-spaces, 2-spaces and 3-spaces are called points, lines and planes. Two subspaces are called disjoint when they have no point in common, i.e., when the intersection has dimension 0.

**Proposition 2.1** *Let  $\Gamma$  be the dual polar graph of type  $D_4(\mathbf{F}_q)$ . Fix elements  $A_0 \in \mathcal{F}_1$  and  $B_0 \in \mathcal{F}_2$  with  $A_0 \sim B_0$ . Let  $\Delta$  be the subgraph of  $\Gamma$  induced on the set of vertices disjoint from  $A_0$  or  $B_0$ . Then  $\Delta$  is distance-regular with intersection array  $\{q^3, q^3 - 1, q^3 - q, q^3 - q^2 + 1; 1, q, q^2 - 1, q^3\}$ .*

The distance distribution diagram is



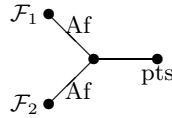
**Proof:** There are  $q^6$  elements  $A \in \mathcal{F}_1$  disjoint from  $A_0$  and the same number of  $B \in \mathcal{F}_2$  disjoint from  $B_0$ , so that  $\Delta$  has  $2q^6$  vertices.

Given  $A \in \mathcal{F}_1$ , there are  $q^3 + q^2 + q + 1$  elements  $B \in \mathcal{F}_2$  incident to it. Of these,  $q^2 + q + 1$  contain the point  $A \cap B_0$  and hence are not vertices of  $\Delta$ . So,  $\Delta$  has valency  $q^3$ .

Two vertices  $A, A' \in \mathcal{F}_1$  have distance 2 in  $\Delta$  if and only if they meet in a line, and the line  $L = A \cap A'$  is disjoint from  $B_0$ . If this is the case, then  $L$  is in  $q + 1$  elements  $B \in \mathcal{F}_2$ , one of which meets  $B_0$ , so that  $A$  and  $A'$  have  $c_2 = q$  common neighbours in  $\Delta$ .

Given vertices  $A \in \mathcal{F}_1$  and  $B \in \mathcal{F}_2$  that are nonadjacent, i.e., that meet in a single point  $P$ , the neighbours  $A'$  of  $B$  at distance 2 to  $A$  in  $\Delta$  correspond to the lines  $L$  on  $P$  in  $A$  disjoint from  $B_0$  and nonorthogonal to the point  $A_0 \cap B$ . There are  $q^2 + q + 1$  lines  $L$  on  $P$  in  $A$ ,  $q + 1$  of which are orthogonal to the point  $A_0 \cap B$ , and one further of which meets  $B_0$ . (Note that the points  $A_0 \cap B$  and  $A \cap B_0$  are nonorthogonal since neither point is in the plane  $A_0 \cap B_0$  and  $V$  does not contain totally isotropic 5-spaces.) It follows that  $c_3 = q^2 - 1$ , and also that  $\Delta$  has diameter 4, and is distance-regular.  $\square$

The geometry induced by the incidence relation of  $D_4(F)$  on the vertices of  $\Delta$ , together with the points and lines contained in the planes disjoint from  $A_0 \cup B_0$ , has Buekenhout-Tits diagram (cf. [P])



that is, the residue of an object  $A \in \mathcal{F}_1$  is an affine 3-space, where the objects incident to  $A$  in  $\mathcal{F}_2$  play the rôle of points. Similar things hold more generally for  $D_n(F)$  with arbitrary  $n$ , and even more generally for all diagrams of spherical type. See also [BB], Theorem 6.1.

Let  $P$  be a nonsingular point, and let  $\phi$  be the reflection in the hyperplane  $H = P^\perp$ . Then  $\phi$  is an element of order two of the orthogonal group that fixes  $H$  pointwise, and consequently interchanges  $\mathcal{F}_1$  and  $\mathcal{F}_2$ . For each  $A \in \mathcal{F}_1$  we

have  $\phi(A) \sim A$ . The quotient  $\Gamma/\phi$  is the dual polar graph of type  $B_3(q)$ , and we see that more generally the dual polar graph of type  $D_{m+1}(q)$  is the extended bipartite double of the dual polar graph of type  $B_m(q)$ . The quotient  $\Delta/\phi$  is a new distance-regular graph discussed in the next section. It is the subgraph consisting of the vertices at maximal distance from a given point in the dual polar graph of type  $B_3(q)$ . For even  $q$  we have  $B_3(q) = C_3(q)$ , and it follows that the symmetric bilinear forms graph on  $\mathbf{F}_q^3$  is distance-regular, see [BCN] Proposition 9.5.10 and the diagram there on p. 286.

### 3 Far from a point in the dual polar graph of type $B_3(q)$

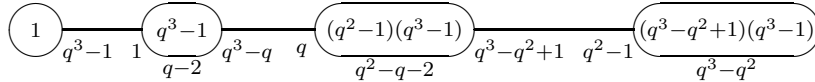
First a very explicit version of the graph of this section.

**Proposition 3.1** (i) Let  $W$  be a vector space of dimension 3 over the field  $\mathbf{F}_q$ , provided with an outer product  $\times$ . Let  $Z$  be the graph with vertex set  $W \times W$  where  $(u, u') \sim (v, v')$  if and only if  $(u, u') \neq (v, v')$  and  $u \times v + u' - v' = 0$ . Then  $Z$  is distance-regular of diameter 3 on  $q^6$  vertices. It has intersection array  $\{q^3 - 1, q^3 - q, q^3 - q^2 + 1; 1, q, q^2 - 1\}$  and eigenvalues  $q^3 - 1, q^2 - 1, -1, -q^2 - 1$  with multiplicities  $1, \frac{1}{2}q(q+1)(q^3-1), (q^3 - q^2 + 1)(q^3 - 1), \frac{1}{2}q(q-1)(q^3-1)$ , respectively.

(ii) The extended bipartite double  $\hat{Z}$  of  $Z$  is distance-regular with intersection array  $\{q^3, q^3 - 1, q^3 - q, q^3 - q^2 + 1; 1, q, q^2 - 1, q^3\}$  and eigenvalues  $\pm q^3, \pm q^2, 0$  with multiplicities  $1, q^2(q^3 - 1), 2(q^3 - q^2 + 1)(q^3 - 1)$ , respectively.

(iii) The distance-1-or-2 graph  $Z_1 \cup Z_2$  of  $Z$ , which is the halved graph of  $\hat{Z}$ , is strongly regular with parameters  $(v, k, \lambda, \mu) = (q^6, q^2(q^3 - 1), q^2(q^2 + q - 3), q^2(q^2 - 1))$ .

The distance distribution diagram of  $Z$  is



**Proof:** Note that the adjacency relation is symmetric, so that  $Z$  is an undirected graph. The computation of the parameters is completely straightforward. Clearly,  $Z$  has  $q^6$  vertices. For  $a, b \in W$  the maps  $(u, u') \mapsto (u+a, u'+(a \times u)+b)$  are automorphisms of  $Z$ , so  $\text{Aut}(Z)$  is vertex-transitive.

The  $q^3 - 1$  neighbours of  $(0, 0)$  are the vertices  $(v, 0)$  with  $v \neq 0$ . The common neighbours of  $(0, 0)$  and  $(v, 0)$  are the vertices  $(cv, 0)$  for  $c \in \mathbf{F}_q, c \neq 0, 1$ . Hence  $a_1 = q - 2$ .

The  $(q^3 - 1)(q^2 - 1)$  vertices at distance 2 from  $(0, 0)$  are the vertices  $(u, u')$  with  $u, u' \neq 0$  and  $u' \perp u$ . The common neighbours of  $(0, 0)$  and  $(u, u')$  are the  $(v, 0)$  with  $v \times u = u'$ , and together with  $(v, 0)$  also  $(v + cu, 0)$  is a common neighbour, so  $c_2 = q$ . Vertices  $(u, u')$  and  $(v, v')$ , both at distance 2 from  $(0, 0)$  are adjacent when  $0 \neq v \perp u'$  and  $v \neq u$  and  $v \times u \neq u'$  and  $v' = u \times v + u'$ , so that  $a_2 = q^2 - q - 2$ .

The remaining  $(q^3 - 1)(q^3 - q^2 + 1)$  vertices have distance 3 to  $(0, 0)$ . They are the  $(w, w')$  with  $w \not\perp w'$  or  $w = 0 \neq w'$ . The neighbours  $(u, u')$  of  $(w, w')$  that lie at distance 2 to  $(0, 0)$  satisfy  $0 \neq u \perp w'$  and  $(0 \neq u')u' = w \times u + w'$ ,

so that  $c_3 = q^2 - 1$ . This shows that  $Z$  is distance-regular with the claimed parameters. The spectrum follows.

The fact that the extended bipartite double is distance-regular, and has the stated intersection array, follows from [BCN], Theorem 1.11.2(vi).

The fact that  $Z_3$  is strongly regular follows from [BCN], Proposition 4.2.17(ii) (which says that this happens when  $Z$  has eigenvalue  $-1$ ).  $\square$

For  $q = 2$ , the graphs here are (i) the folded 7-cube, (ii) the folded 8-cube, (iii) the halved folded 8-cube. All are distance-transitive. For  $q > 2$  these graphs are not distance-transitive.

When  $q$  is a power of two, the graphs  $\hat{Z}$  have the same parameters as certain Kasami graphs, but for  $q > 2$  these are nonisomorphic.

Next, a more geometric description of this graph.

Let  $H$  be a vector space of dimension 7 over the field  $\mathbf{F}_q$ , provided with a nondegenerate quadratic form. Let  $\Gamma$  be the graph of which the vertices are the maximal totally isotropic subspaces of  $H$  (of dimension 3), where two vertices are adjacent when their intersection has dimension 2. This graph is known as the dual polar graph of type  $B_3(q)$ . It is distance-regular with intersection array  $\{q(q^2 + q + 1), q^2(q + 1), q^3; 1, q + 1, q^2 + q + 1\}$ . (See [BCN], §9.4.)

**Proposition 3.2** *Let  $\Gamma$  be the dual polar graph of type  $B_3(q)$ . Fix a vertex  $\pi_0$  of  $\Gamma$ , and let  $\Delta$  be the subgraph of  $\Gamma$  induced on the collection of vertices disjoint from  $\pi_0$ . Then  $\Delta$  is isomorphic to the graph  $Z$  of Proposition 3.1. Its extended bipartite double  $\hat{\Delta}$  (or  $\hat{Z}$ ) is isomorphic to the graph of Proposition 2.1.*

**Proof:** Let  $V$  be a vector space of dimension 8 over  $\mathbf{F}_q$  (with basis  $\{e_1, \dots, e_8\}$ ), provided with the nondegenerate quadratic form  $Q(x) = x_1x_5 + x_2x_6 + x_3x_7 + x_4x_8$ . The point  $P = (0, 0, 0, 1, 0, 0, 0, -1)$  is nonisotropic, and  $P^\perp$  is the hyperplane  $H$  defined by  $x_4 = x_8$ . Restricted to  $H$  the quadratic form becomes  $Q(x) = x_1x_5 + x_2x_6 + x_3x_7 + x_4^2$ .

The  $D_4$ -geometry on  $V$  has disjoint maximal totally isotropic subspaces  $E = \langle e_1, e_2, e_3, e_4 \rangle$  and  $F = \langle e_5, e_6, e_7, e_8 \rangle$ . Fix  $E$  and consider the collection of all maximal totally isotropic subspaces disjoint from  $E$ . This is precisely the collection of images  $F_A$  of  $F$  under matrices  $\begin{pmatrix} I & A \\ 0 & I \end{pmatrix}$ , where  $A$  is alternating with zero diagonal (cf. [BCN], Proposition 9.5.1(i)). Hence, we can label the  $q^6$  vertices  $F_A \cap H$  of  $\Delta$  with the  $q^6$  matrices  $A$ .

Two vertices are adjacent when they have a line in common, that is, when they are the intersections with  $H$  of maximal totally isotropic subspaces in  $V$ , disjoint from  $E$ , that meet in a line contained in  $H$ . Let

$$A = \begin{pmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{pmatrix}.$$

Then  $\det A = (af - be + cd)^2$ , and if  $\det A = 0$  but  $A \neq 0$ , then  $\ker A$  has dimension 2, and is spanned by the four vectors  $(0, f, -e, d)^\top$ ,  $(-f, 0, c, -b)^\top$ ,  $(e, -c, 0, a)^\top$ ,  $(-d, b, -a, 0)^\top$ . Writing the condition that matrices  $A$  and  $A'$  belong to adjacent vertices we find the description of Proposition 3.1 if we take  $u = (c, e, f)$  and  $u' = (-d, b, -a)$ .  $\square$

## 4 History

In 1991 the second author constructed the graphs from Section 2 and the first author those from Section 3. Both were mentioned on the web page [ac], but not published thus far. These graphs have been called the Pasechnik graphs and the Brouwer-Pasechnik graphs, respectively, by on-line servers.

## References

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